Basic Mathematics

Introduction to Matrices

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic understanding of matrices, their addition and subtraction and elementary row operations.

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1. Matrices (Introduction)

A matrix is a rectangular array of numbers.

Example 1 Each of the following are examples of matrices.

$$A = \begin{pmatrix} 4 & 3 & -1 \\ 8 & -0.5 & 34 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & -3 & 3 & -7 \end{pmatrix},$$

$$C = \begin{pmatrix} 5 & -3 & 7 \\ 7 & 0 & -7 \\ 0 & 25 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} -5 \\ 4 \\ -57 \\ 34 \end{pmatrix}$$

The matrix A has two rows and three columns, it is a 2×3 (read as "two by three"), matrix.
The matrix B has one row and four columns, it is a 1×4 matrix.
The matrix C has three rows and three columns, it is a 3×3 matrix.
The matrix D has four rows and one column, it is a 4×1 matrix.

The matrix B is also called a row vector whilst the matrix D is called a column vector.

The elements of a matrix are referred to (or labelled) by their row number and column number, but *always in that order*, i.e. the row number followed by the column number. The ij element of a matrix X, often written as x_{ij}, is the element in the i-th row and j-th column.

Example 2 The matrix X is written below.

(a) What is the element x_{21}?
(b) What is the element x_{32}?
(c) What is the label for 5?
(d) What is the label for 9?

$$X = \begin{pmatrix} -3 & 4 & 7 & -2 \\ -4 & -6 & 21 & 5 \\ -7 & 0 & 8 & 9 \end{pmatrix}$$

Solution

(a) The element x_{21} is -4.
(b) The element x_{32} is 0.
(c) 5 is the element x_{24}.
(d) 9 is the element x_{34}.

Quiz What is the label for -7 in the matrix C of **example 1**?

(a) c_{32}, (b) c_{23}, (c) c_{13}, (d) c_{21}.

2. Addition of Matrices

Two matrices, A and B, may be added together provided that they have the same number of rows and the same number of columns. If A and B are both $m \times n$ matrices then $A + B$ exists and is also a $m \times n$ matrix. If a_{ij} and b_{ij} are the ij elements of A and B respectively, then $a_{ij} + b_{ij}$ is the ij element of $A + B$.

Example 3 If the matrices A and B are

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 4 & 5 \\ -1 & 5 & -7 \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} 1+3 & 2+4 & 3+5 \\ 0+(-1) & -1+5 & 2+(-7) \end{pmatrix} = \begin{pmatrix} 4 & 6 & 8 \\ -1 & 4 & -5 \end{pmatrix}.$$

Similarly

$$A - B = \begin{pmatrix} 1-3 & 2-4 & 3-5 \\ 0-(-1) & -1-5 & 2-(-7) \end{pmatrix} = \begin{pmatrix} -2 & -2 & -2 \\ 1 & -6 & 9 \end{pmatrix}.$$

Exercise 1. The matrices X, Y, Z are given below.

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \; Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, \; Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

Find the following matrices. (Click on the green letters for solutions.)

(a) $X + Y$, (b) $X - Y$, (c) $Y + Z$,
(d) $(X + Y) + Z$, (e) $X + (Y + Z)$, (f) $X - (Y - Z)$.

Parts (d) and (e) above provide an example of the general rule that

$$(X + Y) + Z = X + (Y + Z)\,.$$

NB. All of the *rules for brackets* in the addition of matrices are exactly the same as the corresponding rule for numbers.

Quiz From exercise 1, what is the 23 element of $X - (Y - Z)$?

(a) -6, (b) -2, (c) 6, (d) 10.

If a matrix X is multiplied by a number λ, the resulting matrix, λX, is X with every element multiplied by λ. This is called *scalar multiplication*. (*Multiplying matrices* is the subject of a separate package.)

Example 4 If $\lambda = 2$ and

$$X = \begin{pmatrix} -1 & 3 \\ 5 & 2 \end{pmatrix} \quad \text{then} \quad \lambda X = 2\begin{pmatrix} -1 & 3 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 6 \\ 10 & 4 \end{pmatrix}.$$

The *rules for brackets and scalar multiplication* of matrices are the same as for multiplication and addition of numbers.

Exercise 2. The matrices X, Y, Z are given below.

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \; Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, \; Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

Write out the following matrices. (Click on the green letters for the solutions.)

(a) $2(X+Y)$, (b) $2X+2Y$, (c) $2Y+Z$,
(d) $2(X+Y)+Z$, (e) $2X-3(Y-Z)$, (f) $2X-3Y+3Z$.

3. The Transpose of a Matrix

The transpose of a matrix A, written A^T, is the matrix obtained by writing the rows of A as the columns of A^T.

Example 5

$$\text{If } A = \begin{pmatrix} -1 & 2 & 3 \\ 4 & 0 & -7 \end{pmatrix} \quad \text{then} \quad A^T = \begin{pmatrix} -1 & 4 \\ 2 & 0 \\ 3 & -7 \end{pmatrix}.$$

EXERCISE 3. Write down the transpose of each of the following matrices. (Click on the green letters for the solutions.)

(a) $\begin{pmatrix} 1 & 0 \\ -2 & 4 \\ 5 & -4 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 4 \\ 6 & 0 & 5 \end{pmatrix}$, (c) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$,

Quiz If A is a $m \times n$ matrix and $A = A^T$, which of the following must *always* be true?

(a) m and n may be different, (b) $m = n$, (c) A^T is $m \times n$.

4. Row (and Column) Operations

There are three basic *row operations* which may be performed on matrices. These are:

(1) Multiplication of any row by a non-zero number.
(2) Interchange of two rows.
(3) Addition of a multiple of one row to another.

The following notation will be used for these operations.

(1) $R_3 \to 4R_3$ means row 3 is changed to 4×row 3.
(2) $R_1 \leftrightarrow R_2$ means row 1 is interchanged with row 2.
(3) $R_2 \to R_2 + 3R_3$ means row 2 has 3×row 3 added to it.

Definition If a matrix B is obtained from A after a *sequence* of these row operations, then they are said to be *row equivalent*.
In a similar manner, *column operations* may be performed on a matrix to give a matrix which is *column equivalent* to it. Column operations performed on A are row operations performed on A^T.

Example 6

Perform the following row operations on the adjacent matrix X.

(1) $R_1 \leftrightarrow R_2$. **(2)** $R_2 \to -2R_2$.

(3) $R_1 \to R_1 + 2R_2$.

$$X = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix}$$

Solution

$$\textbf{(1)} \quad X = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & 3 & 2 \\ 1 & 2 & 0 \\ 4 & 5 & 3 \end{pmatrix}.$$

$$\textbf{(2)} \quad X = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix} \xrightarrow{R_2 \to 2R_2} \begin{pmatrix} 1 & 2 & 0 \\ -2 & 6 & 4 \\ 4 & 5 & 3 \end{pmatrix}.$$

$$\textbf{(3)} \quad X = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix} \xrightarrow{R_1 \to R_1 + 2R_2} \begin{pmatrix} -1 & 8 & 4 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix}.$$

Example 7

Perform the following *sequence* of row operations on the adjacent matrix X. $R_2 \leftrightarrow R_3$, followed by $R_2 \rightarrow -2R_2$, followed by $R_1 \rightarrow R_1 + 2R_3$.

$$X = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix}$$

Solution In this case, the first operation is performed on X to obtain X_1 and then the second operation is performed on X_1, to obtain a matrix X_2 say, etc.

$$X = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 5 & 3 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & 3 \\ -1 & 3 & 2 \end{pmatrix}$$

$$\xrightarrow{R_2 \rightarrow -2R_2} \begin{pmatrix} 1 & 2 & 0 \\ -8 & -10 & -6 \\ -1 & 3 & 2 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 + 2R_3} \begin{pmatrix} -1 & 8 & 4 \\ -8 & -10 & -6 \\ -1 & 3 & 2 \end{pmatrix}$$

The final matrix is said to be *row equivalent* to X.

Exercise 4.

Perform the following *sequence* of row operations on the matrix Y. The resulting matrix, Y', is said to be *upper triangular*. (Click on the green letters for the solutions.)

$$Y = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix}$$

(a) $R_1 \to \frac{1}{2}R_1$, (b) $R_2 \to R_2 - R_1$, (c) $R_3 \to R_3 - 2R_1$.

Exercise 5.

Perform the following *sequence* of row operations on the matrix Z. The resulting matrix, Z', is the *3 by 3 identity* matrix. (Click on the green letters for the solutions.)

$$Z = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) $R_2 \to R_2 - 2R_3$, (b) $R_1 \to R_1 - R_3$, (c) $R_1 \to R_1 - 3R_2$.

5. Quiz on Matrices

The following questions all refer to the matrix A.

$$A = \begin{pmatrix} -2 & 4 & 1 \\ 3 & 5 & 7 \\ 0 & 1 & -1 \end{pmatrix}$$

Begin Quiz

1. The element a_{32} is
(a) 3, (b) 1, (c) 7, (d) 4.

2. If $X = A^T$ then x_{32} is
(a) 3, (b) 1, (c) 7, (d) 4.

3. Perform the following *sequence* of row operations on the matrix A: $R_1 \to R_1 + R_2$ followed by $R_1 \to R_1 + 8R_3$. The 1×3 element of the resulting matrix is
(a) 3, (b) 9, (c) 0, (d) 5.

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) If the matrices X and Y are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \quad Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix},$$

then

$$X + Y = \begin{pmatrix} 0+(-3) & -1+(-2) & 3+1 \\ 2+(-1) & -1+1 & 4+(-2) \end{pmatrix} = \begin{pmatrix} -3 & -3 & 4 \\ 1 & 0 & 2 \end{pmatrix}.$$

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Exercise 1(b) If the matrices X and Y are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \quad Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix},$$

then

$$X - Y = \begin{pmatrix} 0-(-3) & -1-(-2) & 3-1 \\ 2-(-1) & -1-1 & 4-(-2) \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 3 & -2 & 6 \end{pmatrix}.$$

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Exercise 1(c) If the matrices Y and Z are

$$Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, \qquad Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix},$$

then

$$Y + Z = \begin{pmatrix} -3+3 & -2+(-3) & 1+1 \\ -1+1 & 1+2 & -2+4 \end{pmatrix} = \begin{pmatrix} 0 & -5 & 2 \\ 0 & 3 & 2 \end{pmatrix}.$$

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Exercise 1(d) The three matrices X , Y and Z are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \; Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, \; Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

From part (a),

$$X + Y = \begin{pmatrix} -3 & -3 & 4 \\ 1 & 0 & 2 \end{pmatrix}.$$

Now add the resulting matrix $(X + Y)$ to the matrix Z.

$$(X + Y) + Z = \begin{pmatrix} -3+3 & -3+(-3) & 4+1 \\ 1+1 & 0+2 & 2+4 \end{pmatrix} = \begin{pmatrix} 0 & -6 & 5 \\ 2 & 2 & 6 \end{pmatrix}.$$

Thus

$$(X + Y) + Z = \begin{pmatrix} 0 & -6 & 5 \\ 2 & 2 & 6 \end{pmatrix}.$$

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Exercise 1(e) Given the same three matrices X , Y and Z

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \; Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, \; Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix},$$

sum up them in the order, $X + (Y + Z)$. From part (c),

$$Y + Z = \begin{pmatrix} 0 & -5 & 2 \\ 0 & 3 & 2 \end{pmatrix}.$$

Adding the matrix X to the matrix $(Y + Z)$ yields

$$X + (Y + Z) = \begin{pmatrix} 0+0 & -5+(-1) & 3+2 \\ 2+0 & -1+3 & 4+2 \end{pmatrix} = \begin{pmatrix} 0 & -6 & 5 \\ 2 & 2 & 6 \end{pmatrix}.$$

Comparing this with the previous exercise for $(X + Y) + Z$ it is seen that

$$(X + Y) + Z = X + (Y + Z).$$

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Exercise 1(f) Consider three matrices X , Y and Z

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix} ,\; Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix} ,\; Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix} .$$

To find the matrix $X - (Y - Z)$, first calculate $(Y - Z)$.

$$Y - Z = \begin{pmatrix} -3-3 & -2-(-3) & 1-1 \\ -1-1 & 1-2 & -2-4 \end{pmatrix} = \begin{pmatrix} -6 & 1 & 0 \\ -2 & -1 & -6 \end{pmatrix} .$$

Now subtract the resulting matrix $(Y - Z)$ from the matrix X

$$X-(Y-Z) = \begin{pmatrix} 0-(-6) & -1-1 & 3-0 \\ 2-(-2) & -1-(-1) & 4-(-6) \end{pmatrix} = \begin{pmatrix} 6 & -2 & 3 \\ 4 & 0 & 10 \end{pmatrix}$$

Thus

$$X - (Y - Z) = \begin{pmatrix} 6 & -2 & 3 \\ 4 & 0 & 10 \end{pmatrix} .$$

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Exercise 2(a) The matrices X and Y are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \qquad Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}.$$

To find the matrix $2(X+Y)$, first calculate the sum $(X+Y)$.

$$X+Y = \begin{pmatrix} 0+(-3) & -1+(-2) & 3+1 \\ 2+(-1) & -1+1 & 4+(-2) \end{pmatrix} = \begin{pmatrix} -3 & -3 & 4 \\ 1 & 0 & 2 \end{pmatrix}$$

Scalar multiplication of this matrix by 2 gives

$$2 \times (X+Y) = \begin{pmatrix} 2\times(-3) & 2\times(-3) & 2\times 4 \\ 2\times 1 & 2\times 0 & 2\times 2 \end{pmatrix} = \begin{pmatrix} -6 & -6 & 8 \\ 2 & 0 & 4 \end{pmatrix}.$$

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Exercise 2(b) If the matrices X and Y are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \qquad Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}.$$

then $2\,X$ and $2\,Y$ are

$$2\,X = \begin{pmatrix} 0 & -2 & 6 \\ 4 & -2 & 8 \end{pmatrix}, \qquad 2\,Y = \begin{pmatrix} -6 & -4 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

The sum of these matrices is

$$2X+2Y = \begin{pmatrix} 0+(-6) & -2+(-4) & 6+2 \\ 4+(-2) & -2+2 & 8+(-4) \end{pmatrix} = \begin{pmatrix} -6 & -6 & 8 \\ 2 & 0 & 4 \end{pmatrix}.$$

Comparing this result with the matrix $2\,(X+Y)$ from part (a) shows that

$$2\,(X+Y) = 2\,X + 2\,Y\,.$$

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Exercise 2(c) To find the matrix $2\,Y+Z$, where Y and Z are given by

$$Y=\begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, \qquad Z=\begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix},$$

first calculate $2\,Y$.

$$2\,Y=\begin{pmatrix} 2\times(-3) & 2\times(-2) & 2\times 1 \\ 2\times(-1) & 2\times 1 & 2\times(-2) \end{pmatrix}=\begin{pmatrix} -6 & -4 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

Now add this matrix to the matrix Z to obtain

$$2Y+Z=\begin{pmatrix} -6+3 & -4+(-3) & 2+1 \\ -2+1 & 2+2 & -4+4 \end{pmatrix}=\begin{pmatrix} -3 & -7 & 3 \\ -1 & 4 & 0 \end{pmatrix}.$$

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Exercise 2(d) To calculate the matrix $2\,(X+Y)+Z$, where the matrices X , Y and Z are

$$X=\begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix},\ Y=\begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix},\ Z=\begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

first note that the matrix $2(X+Y)$ was found in part (a), i.e.

$$2\,(X+Y)=\begin{pmatrix} -6 & -6 & 8 \\ 2 & 0 & 4 \end{pmatrix}.$$

Now add this matrix to the matrix Z to obtain

$$2\,(X+Y)+Z=\begin{pmatrix} -6+3 & -6+(-3) & 8+1 \\ 2+1 & 0+2 & 4+4 \end{pmatrix}=\begin{pmatrix} -3 & -9 & 9 \\ 3 & 2 & 8 \end{pmatrix}.$$

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Exercise 2(e) The matrices X , Y and Z are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix},\ Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix},\ Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

To obtain $2\,X - 3\,(Y - Z)$, first calculate $2\,X$ and $3(Y - Z)$ separately.

$$\begin{aligned} 2\,X &= \begin{pmatrix} 0 & -2 & 6 \\ 4 & -2 & 8 \end{pmatrix} \\ 3\,(Y - Z) &= \begin{pmatrix} 3\times(-3-3) & 3\times(-2-(-3)) & 3\times(1-1) \\ 3\times(-1-1) & 3\times(1-2) & 3\times(-2-4) \end{pmatrix} \\ &= \begin{pmatrix} -18 & 3 & 0 \\ -6 & -3 & -18 \end{pmatrix} \end{aligned}$$

Now subtract $3\,(Y - Z)$ from $2\,X$ to obtain

$$2\,X - 3\,(Y - Z) = \begin{pmatrix} 18 & -5 & 6 \\ 10 & 1 & 26 \end{pmatrix}.$$

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Exercise 2(f) The matrices X, Y and Z are

$$X = \begin{pmatrix} 0 & -1 & 3 \\ 2 & -1 & 4 \end{pmatrix}, \; Y = \begin{pmatrix} -3 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix}, \; Z = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

To find $C = 2\,X - 3\,Y + 3\,Z$, first find $2\,X$, $3\,Y$ and $3\,Z$:

$$2X = \begin{pmatrix} 0 & -2 & 6 \\ 4 & -2 & 8 \end{pmatrix}, 3Y = \begin{pmatrix} -9 & -6 & 3 \\ -3 & 3 & -6 \end{pmatrix}, 3Z = \begin{pmatrix} 9 & -9 & 3 \\ 3 & 6 & 12 \end{pmatrix}.$$

The matrix C is now

$$\begin{aligned} C &= \begin{pmatrix} 0-(-9)+9 & -2-(-6)+(-9) & 6-3+3 \\ 4-(-3)+3 & -2-3+6 & 8-(-6)+12 \end{pmatrix} \\ &= \begin{pmatrix} 18 & -5 & 6 \\ 10 & 1 & 26 \end{pmatrix}. \end{aligned}$$

Comparing this with the result of Exercise 2(e) it is seen that

$$2\,X - 3\,(Y - Z) = 2\,X - 3\,Y + 3\,Z\,.$$

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Exercise 3(a)
If the 3×2 matrix A is

$$A = \begin{pmatrix} 1 & 0 \\ -2 & 4 \\ 5 & -4 \end{pmatrix},$$

then its transpose is the 2×3 matrix, A^T, whose columns are the rows of the matrix A:

$$A^T = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 4 & -4 \end{pmatrix}.$$

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Exercise 3(b)
If the matrix B is the 3×3 matrix

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 4 \\ 6 & 0 & 5 \end{pmatrix},$$

then its transpose matrix, B^T, is also a 3×3 matrix but the rows of B are the columns of B^T. Thus

$$B^T = \begin{pmatrix} 1 & 3 & 6 \\ 2 & -1 & 0 \\ 3 & 4 & 5 \end{pmatrix}.$$

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Exercise 3(c)

If a matrix X has a single row

$$X = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix},$$

then its transpose matrix, X^T, is a matrix with a single column

$$X^T = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The transpose of a row matrix is a column matrix.

Matrices such as X are sometimes referred to as row vectors and matrices such as X^T are sometimes called column vectors.

Click on the green square to return □

Exercise 4(a) If Y is the matrix

$$Y = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix},$$

then performing the row operation $R_1 \to \frac{1}{2}R_1$, we obtain the matrix

$$Y_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix}.$$

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Exercise 4(b) After the first row operation $R_1 \to \frac{1}{2}R_1$ the matrix Y transformed into the matrix Y_1

$$Y = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix} \quad \xrightarrow{R_1 \to \frac{1}{2}R_1} \quad Y_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix}.$$

Now applying the second row operation $R_2 \to R_2 - R_1$ to Y_1 gives

$$Y_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 7 \end{pmatrix}.$$

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Exercise 4(c) After performing the two subsequent row operations the matrix Y was transformed into the matrix Y_2:

$$Y = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix} \xrightarrow{R_1 \to \frac{1}{2} R_1} Y_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 4 & 7 \end{pmatrix}$$

$$\xrightarrow{R_2 \to R_2 - R_1} Y_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 7 \end{pmatrix} .$$

Now applying the row operation $R_3 \to R_3 - 2R_1$ to Y_2 we arrive at the row equivalent matrix Y' where

$$Y' = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} ,$$

with entries below the diagonal which are all zero. Matrices of this type are called the *upper triangular matrices*.

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Exercise 5(a) Performing the row operation $R_2 \to R_2 - 2R_3$ on Z gives

$$Z = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_3} Z_1 = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Exercise 5(b) After the first row operation $R_2 \to R_2 - 2R_3$ the matrix Z was transformed into the matrix Z_1

$$Z = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad \xrightarrow{R_2 \to R_2 - 2R_3} \quad Z_1 = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Performing now the row operation $R_1 \to R_1 - R_3$, we have

$$Z_2 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Exercise 5(c) As a result of the two subsequent row operations the matrix Z was transformed into the matrix Z_2:

$$Z = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \overset{R_2 \to R_2 - 2R_3}{\longrightarrow} Z_1 = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overset{R_1 \to R_1 - R_3}{\longrightarrow} Z_2 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Now applying the row operation $R_1 \to R_1 - 3R_2$ to Z_2 gives

$$Z' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The resulting matrix is the 3×3 *identity* matrix: the elements along its diagonal are $z'_{ii} = 1$ and all other elements are zero.

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Solutions to Quizzes

Solution to Quiz: The matrix is

$$C = \begin{pmatrix} 5 & -3 & 7 \\ 7 & 0 & -7 \\ 0 & 25 & 0 \end{pmatrix}$$

and -7 is in the second row and third column, so it is c_{23}, i.e. the 2×3 element. End Quiz

Solution to Quiz:
From exercise 1(f), the matrix $C = X - (Y - Z)$ is

$$C = X - (Y - Z) = \begin{pmatrix} 6 & -2 & 3 \\ 4 & 0 & 10 \end{pmatrix}.$$

The 23 element of this matrix is the element in the second row and third column, therefore

$$c_{23} = 10\,.$$

End Quiz

Solution to Quiz: The matrix equality

$$A = B$$

means that

$$\begin{aligned} \text{number of rows of } A &= \text{number of rows of } B \\ \text{number of columns of } A &= \text{number of columns of } B\,, \end{aligned}$$

and every element of one matrix is equal to the corresponding element of the other matrix.

If A is a matrix of order $m \times n$, then the transpose matrix has n-rows and m -columns, i.e. A^T is matrix of order $n \times m$. Therefore the equality

$$A = A^T$$

requires at least that the number of rows is the same as the number of columns:

$$m = n\,.$$

(**N.B.** A matrix which has the same number of rows as columns is called a square matrix.) End Quiz